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COMPLETE RIEMANNIAN MANIFOLDS WITH (f, g, u, v, λ)-STRUCTURE

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Yano and Okumura [10] defined the so-called (f, g, u, v, λ)-structure, studied its fundamental properties and gave a characterization of even-dimensional spheres in terms of this structure. In the intrinsic geometry of (f, g, u, v, λ)-structures, some global properties of manifolds with such a structure have been obtained (cf. [2], [5], [6], [8], [9] and [10]). On the other hand, submanifolds of codimension 2 in an almost Hermitian manifold or in an even-dimensional Euclidean space with canonical Kaehlerian structure, and hypersurfaces of an almost contact metric manifold or of an odd-dimensional sphere with canonical contact structure carry, under certain conditions, an (f, g, u, v, λ)-structure. In the differential geometry of submanifolds of a sphere admitting the induced (f, g, u, v, λ)-structure, several results have been proved (cf. [1], [3], [6], [9] and [10]). The main purposes of the present paper are to prove Theorem 3.1, which are closely related to a theorem due to Nakagawa and Yokote [3], and to show that some known theorems concerning (f, g, u, v, λ)-structure can be proved as consequences of the theorems established in the present paper.

In § 1 we discuss properties of almost product structure in a Riemannian manifold, and prove a lemma on the almost product structure and a theorem on the characterization of product spaces of two spheres, using a theorem due to Obata [4]. In § 2 we prove some lemmas on (f, g, u, v, λ)-stuctures for later use. In § 3 complete Riemannian manifolds admitting an (f, g, u, v, λ)-structure and satisfying certain conditions are discussed, and some theorems are proved. Theorem 3.5 stated in § 3 has been already proved by Nakagawa and Yokote [3] under weaker conditions.

1. Riemannian manifolds with almost product structure

Let there be given an m-dimensional Riemannian manifold (M, g) with metric tensor g, components of g being denoted by g_{ji}. (Manifolds, functions, vector fields and other geometric objects throughout this paper are assumed to be differentiable and of class C^∞. The indices h, i, j, k, l, r, s, t run over the range $\{1, \cdots, m\}$ and the summation convention will be used with respect to these

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indices.) Let there be given in (M, g) a tensor field P_i^h of type $(1, 1)$ satisfying

$$P_s{}^h P_i{}^s = P_i{}^h \,, \tag{1.1}$$

$$P_j{}^t P_i{}^s g_{ts} = P_{ji} \,, \tag{1.2}$$

is equivalent to (2.9) for a hypersurface M of $S^{2n+1}(1)$. In [3] Nakagawa and Yokota have studied hypersurfaces of $S^{2n+1}(1)$ satisfying the condition (2.4).

Under the assumptions in Lemma 2.1 *the set N_1 is a bordered set*. In fact, if we suppose that there is an open subset U contained in N_1, then by means of (2.7) we have $f_{ji} \pm H_{ji} = 0$ in U, because $u_i u^i = 1 - \lambda^2 = 0$ in U and hence $u_i = 0$ in U, which together with (2.7) implies that $f_{ji} = 0$ in U, since f_{ji} is skew-symmetric and H_{ji} is symmetric. This contradicts the fact that f_{ji} is of rank $m - 2$ or m in M. Consequently N_1 is necessarily a bordered set (cf. [3]).

Lemma 2.2. *Assume that in (M, g) with an (f, g, u, v, λ)-structure, λ is not zero almost everywhere*, (2.3), (2.4) *and*

$$S_{jih} = v_j(\nabla_i v_h + \nabla_h v_i - 2\lambda g_{ih}) - v_i(\nabla_j v_h + \nabla_h v_j - 2\lambda g_{jh}) \tag{2.10}$$

hold, and there is a symmetric tensor field H_{ji} of type (0,2) *satisfying* (2.5), *where $S_{jih} = S_{ji}{}^t g_{ht}$. Then in M we have* (2.6), (2.7), (2.8) *and* (2.9).

Proof. (2.7) and (2.8) can be proved by using (2.3) and (2.5). We are now going to prove (2.6). For any (f, g, u, v, λ)-structure we have the identity (cf. [9, (1.11)])

$$\begin{aligned} &v^j[S_{jih} - (f_j{}^t f_{tih} - f_i{}^t f_{tjh})] \\ &\quad = (\nabla_i v_h + \nabla_h v_i) - v_i v^t(\nabla_t v_h + \nabla_h v_t) - \lambda f_i{}^t(\nabla_t u_h + \nabla_h u_t) \\ &\qquad - \lambda^2(\nabla_i v_h - \nabla_h v_i) + (\lambda f_i{}^t - u_i v^t)(\nabla_t u_h - \nabla_h u_t) \,, \end{aligned} \tag{2.11}$$

where $f_{jih} = \nabla_j f_{ih} + \nabla_i f_{hj} + \nabla_h f_{ji}$. Substituting (2.3), (2.4) and (2.5) into (2.11) and using $f_{jih} = 0$ which is a direct consequence of (2.3), we obtain

$$\begin{aligned} v^j S_{jih} = {} & (\nabla_i v_h + \nabla_h v_i) - v_i v^t(\nabla_t v_h + \nabla_h v_t) + 2\lambda^2 f_i{}^t H_{th} \\ & -2\lambda^2 \phi f_{ih} + 2(\lambda f_i{}^t - u_i v^t) f_{th} \,. \end{aligned}$$

On the other hand, transvecting (2.10) with v^j gives

$$v^j S_{jih} = (1 - \lambda^2)(\nabla_i v_h + \nabla_h v_i - 2\lambda g_{ih}) - v_i(\nabla_t v_h + \nabla_h v_t)v^t + 2\lambda v_i v_h \,.$$

Thus using (2.1), from the above two equations we have $\nabla_i v_h + \nabla_h v_i = - 2H_{ht} f_i{}^t + 2\lambda g_{ih} + 2\phi f_{ih}$, which together with (2.4) implies (2.6) in $\overline{N}_0$ and consequently in M. Finally we have (2.9) by substituting (2.6) into (2.4). Thus Lemma 2.2 is proved.

Lemma 2.3. *Under the assumptions in Lemma* 2.1 *we have*

$$H_{jt}v^t = \beta u_j + \gamma v_j \,, \tag{2.14}$$

$$\alpha + \gamma = 2\phi \tag{2.15}$$

in $\bar{N}_1$, *where* $H_i^h = H_{it}g^{ht}$, *and* α, β *and* γ *are functions in* $\bar{N}_1$ *defined by* $(1 - \lambda^2)\alpha = H_{st}u^s u^t$, $(1 - \lambda^2)\beta = H_{st}u^s v^t$ *and* $(1 - \lambda^2)\gamma = H_{st}v^s v^t$ *respectively.*

Proof. Transvecting (2.9) with $f_k{}^i$ gives

$$H_{ts}f_j{}^t f_k{}^s + H_{jk} - H_{jt}u^t u_k - H_{jt}v^t v_k = 2\phi(g_{jk} - u_j u_k - v_j v_k) \,.$$

By taking the skew-symmetric parts of the above equation we obtain

$$(H_{jt}u^t)u_k - (H_{kt}u^t)u_j + (H_{jt}v^t)v_k - (H_{kt}v^t)v_j = 0 \,.$$

Thus transvecting the above equation with u^k and v^k and using (2.1), we have (2.3) and (2.14) respectively, because u_i and v_i do not vanish in $\bar{N}_1$.

Next, transvecting (2.9) with $f^{ji} = g^{jt}f_t{}^i$ and using (2.1), we obtain

$$H_t{}^t = 2\phi(n - (1 - \lambda^2)) + H_{ts}u^t u^s + H_{ts}v^t v^s \,. \tag{2.16}$$

On the other hand, transvecting (2.9) with $u^j v^i$ and using (2.1) yield $\lambda(H_{ts}u^t u^s + H_{ts}v^t v^s) = 2\lambda(1 - \lambda^2)\phi$. Thus we have

$$H_{ts}u^t u^s + H_{ts}v^t v^s = 2(1 - \lambda^2)\phi \tag{2.17}$$

in $\bar{N}_0$ and consequently in M. Restricting (2.17) to $\bar{N}_1$ gives (2.15). Finally by substituting (2.17) into (2.16) we have (2.12) in M. Thus Lemma 2.3 is proved.

Lemma 2.4. *If in Lemma* 2.1 *the tensor* H_{ji} *satisfies the condition*

$$\nabla_k H_{ji} - \nabla_j H_{ki} = 0 \,, \tag{2.18}$$

then we have in $\bar{N}_1$

$$\phi(1 - \beta) = \alpha \,, \tag{2.19}$$

$$v^t \nabla_t \alpha = u^t \nabla_t \beta \,. \tag{2.20}$$

Proof. Differentiating (2.13) covariantly gives

$$(\nabla_k H_{jt})u^t + H_{jt}(\nabla_k u^t) = (\nabla_k \alpha)u_j + (\nabla_k \beta)v_j + \alpha\nabla_k u_j + \beta\nabla_k v_j$$

in $\bar{N}_1$. Taking skew-symmetric parts of both sides of the above equation and using (2.18) we obtain

$$\begin{aligned} H_{jt}(\nabla_k u^t) - H_{kt}(\nabla_j u^t) = {} & (\nabla_k \alpha)u_j - (\nabla_j \alpha)u_k + (\nabla_k \beta)v_j - (\nabla_j \beta)v_k \\ & + \alpha(\nabla_k u_j - \nabla_j u_k) + \beta(\nabla_k v_j - \nabla_j v_k) \,. \end{aligned}$$

Next, if we substitute (2.3), (2.4), (2.6) and (2.7) into the above equation and use (2.9), then we have

$$2\{\phi(1-\beta)-\alpha\}f_{kj}=(\nabla_k\alpha)u_j-(\nabla_j\alpha)u_k+(\nabla_k\beta)v_j-(\nabla_j\beta)v_k\,, \tag{2.21}$$

from which it follows that $\nabla_j\alpha$ and $\nabla_j\beta$ are linear combinations of u_j and v_j, i.e., that

$$\nabla_j\alpha=A_1u_j+A_2v_j\,,\qquad \nabla_j\beta=B_1u_j+B_2v_j\,, \tag{2.22}$$

where A_1, A_2, B_1 and B_2 are certain functions in $\bar{N}_1$. Thus (2.21) reduces to $2\{\phi(1-\beta)-\alpha\}f_{kj}=-(A_2-B_1)(u_kv_j-u_jv_k)$, which implies that $\phi(1-\beta)=\alpha$ and $A_2=B_1$, since f_{ji} is of rank $2n-2\geq 2$ in $\bar{N}_1$ by assumption. Thus we have (2.19) and (2.20), and Lemma 2.4 is proved.

Remark. If (M,g) is a hypersurface of a sphere $S^{2n+1}(1)$, the (f,g,u,v,λ)-structure of (M,g) is the induced one, and H_{ji} is the second fundamental tensor of the hypersurface, then the condition (2.18) is nothing but the structure equation of Codazzi for the immersion of M into $S^{2n+1}(1)$.

Lemma 2.5. *Under the conditions in Lemma* 2.4, *the equation*

$$\begin{aligned}&H_{kt}H_i^{\ t}-2\phi H_{ki}+\{\beta+\phi^2(1+\beta)\}g_{ki}\\&\quad=(1-\lambda^2)^{-1}\beta(\beta+1)(1+\phi^2)(u_ku_i+v_kv_i)\end{aligned} \tag{2.23}$$

holds in $\bar{N}_1$, *and the function* ϕ *is constant in* M.

Proof. Differentiating (2.14) covariantly and using (2.6) and (2.7), we have

$$\begin{aligned}&(\nabla_kH_{jt})v^t+H_{jt}(-H_{ks}f^{ts}+\lambda\delta_k^t)\\&\quad=(\nabla_k\beta)u_j+(\nabla_k\gamma)v_j+\beta(f_{kj}-\lambda H_{kj})+\gamma(-H_{kt}f_j^{\ t}+\lambda g_{kj})\,.\end{aligned}$$

By taking the skew-symmetric parts of the above equation and using (2.9) and (2.18), we obtain

$$\begin{aligned}&-2H_{jt}H_{ks}f^{ts}-2(\beta+\gamma\phi)f_{kj}\\&\quad=(\nabla_k\beta)u_j-(\nabla_j\beta)u_k+(\nabla_k\gamma)v_j-(\nabla_j\gamma)v_k\,.\end{aligned} \tag{2.24}$$

Transvecting (2.24) with v^k gives that $\nabla_j\gamma$ is a linear combination of u_j and v_j, i.e., that

$$\nabla_j\gamma=C_1u_j+C_2v_j\,, \tag{2.25}$$

where C_1 and C_2 are certain functions in $\bar{N}_1$. Using (2.22) and (2.25), we can reduce (2.24) to

$$-2H_{jt}H_{ks}f^{ts}-2(\beta+\phi\gamma)f_{kj}=(B_2-C_1)(v_ku_j-v_ju_k)\,. \tag{2.26}$$

Transvecting (2.26) with $v^k u^j$ gives that $2\lambda(\alpha\gamma - \beta^2 - \beta - \gamma\phi) = (B_2 - C_1) \cdot (1 - \lambda^2)$ in $\bar{N}_1$, which together with (2.15) and (2.19) implies

$$- 2\lambda\beta(1 + \beta)(1 + \phi^2) = (B_2 - C_1)(1 - \lambda^2) \tag{2.27}$$

in $\bar{N}_1$. Substituting (2.27) into (2.26) and using (2.9), we have in $\bar{N}_1$

$$\begin{aligned}(H_{ks}f_t^{\ s} - 2\phi H_{kt})f_j^{\ t} - (\beta + \phi\gamma)f_{kj} \\ = \lambda(1 - \lambda^2)^{-1}\beta(\beta + 1)(1 + \phi^2)(v_j u_k - u_j v_k) .\end{aligned}$$

Transvecting the above equation with $f_i^{\ j}$ and using (2.13), (2.14), (2.15) and (2.19), we obtain (2.23) in $\bar{N}_1$.

Next, we are going to prove that ϕ is constant in M. Let ρ be an eigenvalue of $H_i^{\ h}$ associated with an eigenvector of $H_i^{\ h}$, which is orthogonal to u^h and v^h. Then using (2.23) we see that ρ satisfies the quadrastic equation

$$\rho^2 - 2\phi\rho + \{\beta + \phi^2(1 + \beta)\} = 0 \tag{2.28}$$

in $\bar{N}_1$, which implies that β is nonpositive because ρ is real due to $H_{ji} = H_{ij}$. Differentiating covariantly the second equation of (2.22) yields $\nabla_k \nabla_j \beta = (\nabla_k B_1)u_j + B_1(\nabla_k u_j) + (\nabla_k B_2)v_j + B_2 \nabla_k v_j$. By taking the skew-symmetric parts of this equation and using (2.3) and (2.4), we obtain $(\nabla_k B_1)u_j - (\nabla_j B_1)u_k + (\nabla_k B_2)v_j - (\nabla_j B_2)v_k = 2(B_1 + \phi B_2)f_{jk}$. Since f_{ji} is of rank $2n \geq 4$ in $\bar{N}_1$, we have

$$B_1 + \phi B_2 = 0 \tag{2.29}$$

in $\bar{N}_0 \cap \bar{N}_1$ and consequently in $\bar{N}_1$. If we now differentiate (2.19) covariantly, then we have $\nabla_j \alpha = (1 - \beta)\nabla_j \phi - \phi \nabla_j \beta$, which together with (2.22) implies

$$A_1 u_j + A_2 v_j = (1 - \beta)\nabla_j \phi - \phi(B_1 u_j + B_2 v_j) . \tag{2.30}$$

On the other hand, we have already proved $A_2 = B_1$ (cf. (2.20)) in the proof of Lemma 2.4. Thus using (2.29), (2.30) and $A_2 = B_1$ we find $(1 - \beta)\nabla_j \phi = (A_1 + \phi B_1)u_j$. Since β is nonpositive, we have $1 - \beta \neq 0$, and therefore the above equation becomes $\nabla_j \phi = \tau u_j$, τ being a certain function in $\bar{N}_1$. Differentiating this equation covariantly, taking the skew-symmetric parts, and using (2.3), we obtain $(\nabla_k \tau)u_j - (\nabla_j \tau)u_k + 2\tau f_{kj} = 0$. Since f_{kj} is of rank $2n \geq 4$ in $\bar{N}_1$, $\tau = 0$. Consequently, ϕ is necessarily constant in $\bar{N}_1$ and hence in M. Thus Lemma 2.5 is proved.

Lemma 2.6. *Assume that in Lemma* 2.1 *the tensor field H_{ji} satisfies the condition* (2.18), *and the sectional curvature $K(\theta)$ of (M, g) with respect to the section θ spanned by u^h and v^h is constant in $\bar{N}_1$. Then α, β and γ are all constant and, in particular, $\beta = 0$ or -1. Moreover, we have*

$$T_t^{\ h} T_i^{\ t} = - \beta(1 + \phi^2)\delta_i^h , \tag{2.31}$$

(2.32) $$\nabla_k T_{ji} - \nabla_j T_{ki} = 0 ,$$

where $T_{ji} = H_{ji} - \phi g_{ji}$ *and* $T_j{}^h = T_{jt} g^{ht}$.

Proof. Differentiating (2.7) covariantly and using (2.8) give $\nabla_k \nabla_j u_i = \nabla_k f_{ji} - (H_{kt}u^t - v_k)H_{ji} - \lambda \nabla_k H_{ji}$. By taking the skew-symmetric parts with respect to j and k from this equation and using the Ricci identity and (2.18), we have

$$- K_{kjih}u^h = \nabla_k f_{ji} - \nabla_j f_{ki} - (H_{kt}u^t - v_k)H_{ji} + (H_{jt}u^t - v_j)H_{ki} ,$$

where K_{kjih} are the components of the curvature tensor of (M, g). Transvecting the above equation with v^i and using (2.13) and (2.14), we find .

$$- K_{kjih}v^i u^h = (\nabla_k f_{ji})v^i - (\nabla_j f_{ki})v^i - (\alpha u_k + \beta v_k - v_k)(\beta u_j + \gamma v_j) + (\alpha u_j + \beta v_j - v_j)(\beta u_k + \gamma v_k) ,$$

which reduces to, in consequence of (2.1), (2.6), (2.7) and (2.8),

$$K_{kjih}v^i u^h = (\alpha\gamma - \beta^2 + 1)(v_j u_k - v_k v_j) .$$

Thus the sectional curvature $K(\theta)$ is given by

$$K(\theta) = - K_{kjih}v^k u^j v^i u^h / [(u_i u^i)(v_i v^i)] = \alpha\gamma - \beta^2 + 1 .$$

Since $K(\theta)$ is constant, $\alpha\gamma - \beta^2 + 1$ is also so. Thus α, β and γ are constant because of (2.15) and (2.19).

Since β and γ are constant, we have $B_2 = C_1 = 0$, where B_2 and C_1 are functions appearing in (2.22) and (2.25). Thus using (2.27) we obtain $\beta(\beta + 1) = 0$ in $\bar{N}$ and hence $\beta = 0$ or -1 in M. Substituting $\beta(\beta + 1) = 0$ into (2.23) gives

$$H_{kt}H_i{}^t - 2\phi H_{ki} + \{\beta + \phi^2(1 + \beta)\}g_{ki} = 0 ,$$

which is equivalent to (2.31). Next by means of (2.18) we have (2.32) since ϕ is constant. Hence Lemma 2.6 is proved.

Lemma 2.7. *Let (M, g) be a Riemmannian manifold with an (f, g, u, v, λ)-structure satisfying the conditions in Lemma 2.6. If $\beta = 0$, then $H_{ji} = \phi g_{ji}$. If $\beta = -1$, then the tensor field $P_i{}^h$ of type (1,1) defined by*

(2.33) $$P_i{}^h = \tfrac{1}{2}(1 + \phi^2)^{-1/2}((- \phi + \sqrt{1 + \phi^2})\delta_i^h + H_i{}^h)$$

is an almost product structure of rank n in (M, g) such that

(2.34) $$\nabla_k P_{ji} - \nabla_j P_{ki} = 0 .$$

where $P_{ji} = P_j{}^t g_{it}$.

Proof. First we assume that $\beta = 0$. Then by substituting $\beta = 0$ into (2.31) we have $T_{ji} = 0$, which implies $H_{ji} = \phi g_{ji}$.

Next we assume that $\beta = -1$. Then substituting $\beta = -1$ into (2.31) we find

$$T_t{}^h T_i{}^t = (1 + \phi^2)\delta_i^h . \tag{2.35}$$

On the other hand, using (2.33) and $T_i{}^h = H_i{}^h - \phi\delta_i^h$ we have

$$P_i{}^h = \tfrac{1}{2}(\delta_i^h + (1 + \phi^2)^{-1/2}T_i{}^h) . \tag{2.36}$$

(2.36) and (2.35) imply $P_t{}^h P_i{}^t = P_i{}^h$, which shows that $P_i{}^h$ is an almost product structure. (2.32) and (2.36) imply (2.34). Contracting i and h in (2.33) and using (2.12) we find $P_t{}^t = n$, which means that $P_i{}^h$ is of rank n. Thus Lemma 2.7 is proved.

Lemma 2.8. *Assume that in Lemma* 2.1 *the tensor field H_{ji} satisfies* (2.18), *and the curvature tensor of (M, g) is of the form*

$$K_{kjih} = g_{kh}g_{ji} - g_{jh}g_{ki} + H_{kh}H_{ji} - H_{jh}H_{ki} . \tag{2.37}$$

If the scalar curvature $K = K_{kjih}g^{kh}g^{ji}$ is constant, then α, β and γ are all constant and the same conclusions as those stated in Lemma 2.7 *are valid.*

Proof. From (2.37), we have by contraction

$$K = 2n(2n - 1) + (H_t{}^t)^2 - H_{ts}H^{ts} , \tag{2.38}$$

where $H^{ts} = g^{tj}g^{si}H_{ji}$. On the other hand, from (2.23) we obtain by transvecting with g^{ki}

$$H_{ts}H^{ts} - 2\phi H_t{}^t + 2n\{\beta + \phi^2(1 + \beta)\} = 2\beta(\beta + 1)(1 + \phi^2) . \tag{2.39}$$

Using now (2.12), (2.38) and (2.39), we see that β is constant, since K is constant. Thus from (2.15) and (2.19) it follows that α and γ are also constant, because β and ϕ are constant. Therefore we can derive the same conclusions as stated in Lemma 2.7, and Lemma 2.8 is proved.

Remark. If (M, g) is a hypersurface of a sphere $S^{2n+1}(1)$, the (f, g, u, v, λ)-structure of (M, g) is the induced one, and H_{ji} is the second fundamental tensor of the hypersurface, then (2.37) is nothing but the structure equation of Gauss for the hypersurface.

In the sequel we need the following lemma proved by Nakagawa and Yokote [4].

Lemma 2.9. *Assume that in Lemma* 2.1 *the tensor field H_{ji} satisfies* (2.18), *and the curvature tensor of (M, g) is of the form* (2.37). *If (M, g) is compact, then we have $\beta(\beta + 1) = 0$, that is, $\beta = 0$ or -1 in M.*

3. Complete Riemannian manifolds with an (f, g, u, v, λ)-structure

First, we prove

Theorem 3.1. *Let (M, g) be a complete connected Riemannian manifold of dimension $2n \geq 4$ with an (f, g, u, v, λ)-structure such that λ is not zero almost everywhere in M and that there be given a tensor field H_{ji} of type (0.2) satisfying (2.5), (2.18). Further assume that the (f, g, u, v, λ)-structure satisfies (2.3), (2.4), (2.6) where ϕ is a certain function in M. If the sectional curvature $K(\theta)$ of (M, g) with respect to the section θ spanned by u^h and v^h is constant in $\bar{N}_1$, then the function ϕ is necessarily constant, and (M, g) is isometric to one of the following manifolds:*

$$S^{2n}(r)\ , \quad S^n(r_1) \times S^n(r_2)\ , \quad [S^n(r_1) \times S^n(r_2)]^*\ ,$$

where

$$r^{-2} = 1 + \phi^2\ , \quad r_1^{-2} = 2(1 + \phi^2 + \phi\sqrt{1 + \phi^2})\ ,$$
$$r_2^{-2} = 2(1 + \phi^2 - \phi\sqrt{1 + \phi^2})\ .$$

Moreover, H_{ji} takes the form

$$H_{ji} = \phi g_{ji} \tag{3.1}$$

if (M, g) is isometric to $S^2(r)$, or

$$H_{ji} = 2\sqrt{1 + \phi^2}P_{ji} + (\phi - \sqrt{1 + \phi^2})g_{ji} \tag{3.2}$$

if (M, g) is isometric to $S^n(r_1) \times S^n(r_2)$ or $[S^n(r_1) \times S^n(r_2)]^*$, where $P_i{}^h$ is the almost product structure of rank n determined by the local reducibility of (M, g), $P_{ji} = P_j{}^t g_{it}$ and $\nabla_k P_{ji} = 0$.

Proof. Under the assumptions of this theorem, Lemmas 2.1, 2.3, $\cdots$, 2.7 are all valid. By Lemma 2.6 we have $\beta = 0$ or -1, and therefore we consider the following two cases.

Case I: $\beta = 0$. Using (2.13) and (2.19) with $\beta = 0$ we can reduce (2.8) to $\nabla_i\lambda = \phi u_i - v_i$. Covariant differentiation of this equation gives $\nabla_j\Delta_i\lambda = -\lambda(1 + \phi^2)g_{ji}$, in consequence of (2.6), (2.7) and $H_{ji} = \phi g_{ji}$ due to Lemma 2.7. On the other hand, λ is not constant; otherwise, from (2.8) with $\nabla_i\lambda = 0$ and (2.13) it follows that $\beta = 1$, which contradicts to the assumption. Since (M, g) is complete and connected, by Theorem A we thus see that (M, g) is isometric to $S^n(r)$, where $1/r^2 = 1 + \phi^2$.

Case II: $\beta = -1$. Using (2.13) and (2.19) with $\beta = -1$ we can reduce (2.8) to $\nabla_i\lambda = 2(\phi u_i - v_i)$. Covariant differentiation of this equation and use of (2.6) and (2.7) yields

$$\nabla_j\nabla_i\lambda = 2T_{jt}f_i{}^t - 2\lambda\phi T_{ji} - 2\lambda(1 + \phi^2)g_{ji}\ , \tag{3.3}$$

where T_{ji} is given in Lemma 2.6. From (2.35), (2.36) and (3.3) it follows that

$$(3.4)\qquad \begin{aligned} P_j{}^t P_i{}^s \nabla_t \nabla_s \lambda &= -2\lambda(1+\phi^2+\phi\sqrt{1+\phi^2})P_{ji}\,,\\ Q_j{}^t Q_i{}^s \nabla_t \nabla_s \lambda &= -2\lambda(1+\phi^2-\phi\sqrt{1+\phi^2})Q_{ji}\,, \end{aligned}$$

where $P_i{}^h$ is the almost product structure defined in (M,g) by (2.33) and $Q_j{}^h = \delta_i^h - P_i{}^h$. On the other hand, λ is not constant (see Case I). Thus $\nabla_k P_{ji} = 0$ because of (2.34) and Lemma 1.1.

Since $P_i{}^h$ is of rank n and (M,g) is complete and conncted, taking account of Theorem 1.2 and (3.4) we see that (M,g) is isometric to $S^n(r_1) \times S^n(r_2)$ or $[S^n(r_1) \times S^n(r_2)]^*$. Finally, we obtain (3.2) from (2.33). Thus Theorem 3.1 is proved.

Theorem 3.2. *Let (M,g) be a complete connected Riemannian manifold of dimension $2n \geq 4$ with an (f,g,u,v,λ)-structure such that λ is not zero almost everywhere in M, and there be given in M a tensor field H_{ji} of type* (0,2) *satisfying* (2.5) *and* (2.18). *Assume that the (f,g,u,v,λ)-structure of (M,g) satisfies* (2.3), (2.4) *and* (2.6), *and further that the curvature tensor of (M,g) is given by*

$$(2.37)\qquad K_{kjih} = g_{kh}g_{ji} - g_{jh}g_{ki} + H_{kh}H_{ji} - H_{jh}H_{ki}\,.$$

If the scalar curvature K of (M,g) is constant, then the same conclusions as those stated in Theorem 3.1 *are valid.*

Proof. Under the assumptions in Theorem 3.2, Lemma 2.7 follows from Lemma 2.8. Therefore we can prove Theorem 3.2 in the same way as we prove Theorem 3.1.

Taking account of Lemma 2.9, we can prove the following Theorem 3.3 by the same devices as developed in the proof of Theorem 3.1.

Theorem 3.3. *Let (M,g) be a compact connected Riemannian manifold $2n \geq 4$ with an (f,g,u,v,λ)-structure such that λ is not zero almost everywhere in M, and let there be given in M a tensor field H_{ji} of type* (0,2) *satisfying* (2.5) *and* (2.18). *Assume that the (f,g,u,v,λ)-structure of (M,g) satisfies* (2.3), (2.4) *and* (2.6), *and that the curvature tensor of (M,g) is given by* (2.37). *Then the same conclusions as those stated in Theorem* 3.1 *are valid.*

Theorem 3.4. *The conclusions in Theorem* 3.1 (*resp. Theorem* 3.2, *Theorem* 3.3) *are valid, even if in Theorem* 3.1 (*resp. Theorem* 3.2, *Theorem* 3.3) *the condition* (2.6) *is replaced by*

$$(2.10)\quad S_{jih} = v_j(\nabla_i v_h + \nabla_h v_i - 2\lambda g_{ih}) - v_i(\nabla_j v_h + \nabla_h v_j - 2\lambda g_{jh})\,.$$

Proof. By Lemma 2.2, the conditions (2.3), (2.4) and (2.10) imply (2.6). Thus using Lemmas 2.7, 2.8 and 2.9 we can obtain Theorem 3.4.

By means of Theorems 3.1, 3.2 or 3.4 we can prove the theorem in [2], Theorems 9.1, 9.2 in [7] and Theorem 3.2 in [10]. We now state

Lemma 3.5. *Let (M, g) be a complete connected hypersurface immersed in a sphere $S^{m+1}(1)$ with induced metric g_{ji}, and assume that in (M, g) there is an almost product structure P_i^k of rank p such that $\nabla_j P_i^h = 0$. If the second fundamental tensor H_{ji} of the hypersurface (M, g) takes the form $H_{ji} = aP_{ji} + bQ_{ji}$, and $m - 1 \geq p \geq 1$, where a and b are nonzero constants, $P_{ji} = P_j^t g_{it}$, and $Q_{ji} = g_{ji} - P_{ji}$, then the hypersurface (M, g) is congruent to the hypersurfaces $S^p(r_1) \times S^{m-p}(r_2)$ naturally embedded in $S^{m+1}(1)$, where $1/r_1^2 = 1 + a^2$ and $1/r_2^2 = 1 + b^2$.*

By means of Theorems 3.1, 3.2, and Lemma 3.5 we can prove

Theorem 3.6. *Let (M, g) be a complete connected hypersurface immersed in a sphere $S^{2n+1}(1)$ with induced (f, g, u, v, λ)-structure such that λ is not zero almost everywhere in M. Assume that the induced (f, g, u, v, λ)-structure satisfies the condition $\nabla_j v_i - \nabla_i v_j = 2\phi f_{ji}$, ϕ being a certain function in M. If (M, g) satisfies one of the following conditions:* (i) *(M, g) is compact,* (ii) *the scalar curvature K of (M, g) is constant,* (iii) *the sectional curvature $K(\theta)$ of (M, g) with respect to the section θ spanned by u^h and v^h is constant, then ϕ is necessarily constant and the hypersurface (M, g) is congruent to $S^{2n}(r)$ or $S^n(r_1) \times S^n(r_2)$ naturally embedded in $S^{2n+1}(1)$, where $1/r^2 = 1 + \phi^2$, $1/r_1^2 = 2(1 + \phi^2 + \phi\sqrt{1 + \phi^2})$ and $1/r_2^2 = 2(1 + \phi^2 - \phi\sqrt{1 + \phi^2})$,* (cf. Nakagawa and Yokote [3], [4]).

Bibliography

[1] D. E. Blair, G. D. Ludden & K. Yano, *Hypersurfaces of an odd-dimensional sphere*, J. Differential Geometry **5** (1971) 479–486.

[2] ——. *On the intrinsic geometry* of $S^n \times S^n$, Math. Ann. **194** (1971) 68–77.

[3] H. Nakagawa & I. Yokote, *Compact hypersurfaces in an odd-dimensional sphere*, Kōdai Math. Sem. Rep. **25** (1973) 225–245.

[4] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** (1962) 333–340.

[5] H. Suzuki, *Notes on (f, g, u, v, λ)-structure*, Kōdai Math. Sem. Rep. **25** (1973) 153–162.

[6] K. Yano, *Differential geometry* of $S^n \times S^n$, J. Differential Geometry **8** (1973) 181–206.

[7] K. Yano & S. Ishihara, *Notes on hypersurfaces of an odd-dimensional sphere*, Kōdai Math. Sem. Rtp. **24** (1972) 422–429.

[8] K. Yano & U-Hang Ki, *On quasi-normal (f, g, u, v, λ)-structure*, Kōdai Math. Sem. Rep. **24** (1972) 106–120.

[9] ——, *Manifolds with antinormal (f, g, u, v, λ)-structures*, Kōdai Math. Sem. Rep. **25** (1973) 48–62.

[10] K. Yano & M. Okumura, On *(f, g, u, v, λ)-structures*, Kōdai Math. Sem. Rep. **22** (1970) 401–423.

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